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On the periodic intermediate long wave equation

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Abstract. We discuss some properties of a certain physically interesting nonlinear integro-differential equation with periodic boundary conditions. It is the natural periodic analogue of the intermediate long wave equation, and it provides a periodic analogue of the Benjamin–Ono equation in the appropriate limit. Due to the speciality of the integral operator, the equation admits a Bäcklund transformation, an infinity of motion constants, etc. Two simple periodic solutions are exhibited. Finally we note that the equation may be transformed into more than one kind of bilinear equation.

The so-called intermediate long wave (ILW) equation (Joseph 1977, Kubota *et al* 1978) can be written in the form ($\delta > 0$)

$$u_t + \delta^{-1}u_x + 2uu_x + (Tu)_{xx} = 0 \quad (1)$$

on $-\infty < x < \infty$, where

$$Tu = -\frac{1}{2\delta} \int_{-\infty}^{\infty} \coth \frac{\pi}{2\delta}(x-\xi)u(\xi) d\xi \quad (2)$$

and the integral is evaluated in the principal-value sense. The equation can be solved on $(-\infty, \infty)$ via an inverse scattering transform (IST) (Kodama *et al* 1981), and has the analytical structure associated with such equations (Joseph and Egri 1978, Satsuma *et al* 1979).

The physical derivation of (1) and (2) as a model of the evolution of long internal waves of moderate amplitude assumes that $u(x)$ has a classical Fourier transform, and that u vanishes as $|x| \rightarrow \infty$. Even so, one may ask whether (1), (2) admit spatially periodic solutions. This was done by Joseph and Egri (1978), Chen and Lee (1979), and Nakamura and Matsuno (1980), using formal algebraic methods. Unfortunately, the solutions so obtained either contain errors or are subject to a limitation that was obscured by these formal methods.

Alternatively, one may seek an evolution equation for long internal waves of moderate amplitude that are spatially periodic. Then a derivation similar to the usual one leads to (1), but in the periodic case (2) is replaced with

$$Tu = \frac{1}{2L} \int_{-L}^L \tilde{T}(x-\xi; \delta, L)u(\xi) d\xi \quad (3a)$$

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where

$$\tilde{T}(x; \delta, L) = -\frac{2K}{\pi} \left[Z\left(\frac{Kx}{L}\right) + dn\left(\frac{Kx}{L}\right) cs\left(\frac{Kx}{L}\right) \right]. \quad (3b)$$

(Actually, the physical derivation naturally leads to the Fourier representation of \tilde{T} , given in (3c), which is then transformed into (3b).) This derivation also requires that $\int_{-L}^L u \, dx = 0$, which we always may impose on (1) with (3), because any non-zero mean may be removed by a Galilean transformation: $u' = u + \alpha$, $x' = x + 2\alpha t$. In (3b), K denotes the complete elliptic integral of the first kind, $Z(a)$ is Jacobi's zeta function, and $dn(a)$, $cs(a)$ are Jacobian elliptic functions. These all have modulus k , determined by the condition that $K'(k)/K(k) = \delta/L$, where $K'(k)$ is the associated elliptic integral of the first kind. (All of these functions are discussed by Byrd and Friedman (1971).) The purpose of this note is to discuss some of the mathematical structure of (1) with (3).

An alternative, but very useful, representation of \tilde{T} is its Fourier series,

$$\tilde{T}(x; \delta, L) = i \sum_{n \neq 0} \coth\left(\frac{n\pi\delta}{L}\right) \exp\left(\frac{in\pi x}{L}\right) \quad (3c)$$

so that

$$Tu = i \sum_{n \neq 0} \coth\left(\frac{n\pi\delta}{L}\right) \hat{u}_n \exp\left(\frac{in\pi x}{L}\right) \quad (3d)$$

where $\{\hat{u}_n\}$ are the Fourier coefficients of u . This representation follows from the identities (cf Byrd and Friedman 1971, §§ 905.01 and 908.51)

$$Z(a) = \frac{2\pi}{K} \sum_{m=1}^{\infty} \frac{q^m}{1-q^{2m}} \sin\left(\frac{m\pi a}{K}\right)$$

$$cs(a) \, dn(a) = \frac{\pi}{2K} \cot \frac{\pi a}{2K} - \frac{2\pi}{K} \sum_{m=1}^{\infty} \frac{q^m}{1+q^m} \sin\left(\frac{m\pi a}{K}\right)$$

where $q = \exp(-\pi K'/K) = \exp(-\pi\delta/L)$, and from the formal representation (Gel'fand and Shilov 1964, p 32)

$$\cot \frac{1}{2}\theta = 2 \sum_{m=1}^{\infty} \sin m\theta.$$

The usual operator on $(-\infty, \infty)$ may be recovered simply by replacing the sum in (3d) with an integral and rescaling. Similarly, we may recover (2) from (3) by letting $L \rightarrow \infty$, δ fixed. Then $k \rightarrow 1$, $K \rightarrow \pi L/2\delta$, and one may show that

$$\frac{1}{2L} \tilde{T}(x; \delta, L \rightarrow \infty) \rightarrow \frac{1}{2\delta} \coth \frac{\pi x}{2\delta} \quad (4a)$$

which reproduces (2). On the other hand, for $\delta \rightarrow \infty$, L fixed, we have $k \rightarrow 0$, $K \rightarrow \frac{1}{2}\pi$, and

$$\tilde{T}(x; \delta \rightarrow \infty, L) \rightarrow -\cot \pi x/2L. \quad (4b)$$

This is the well known Hilbert kernel on $(-L, L)$. With this kernel, (1) is the natural periodic extension of the Benjamin-Ono equation (Benjamin 1967, Ono 1975). As one would expect, (1) with (3) reduces to the (periodic) KdV equation if $\delta \rightarrow 0$, L fixed.

If $u(x)$ is periodic with period $2L$ and with zero mean, then (Tu) according to (2) and to (3) are identical. That this is so may be seen by rewriting (2) in its Fourier transform representation, and recalling that the FT of a periodic function is a sum of Dirac delta

functions (Gel'fand and Shilov 1964). Thus (1) with (3) may be regarded simply as the most natural way to write the ILW equation when periodic solutions are of interest.

The operator T given in (3) is the most general periodic operator we have found which satisfies the 'T conditions':

$$T(uTv + vTu) = TuTv - uv \tag{T1}$$

$$\int_{-\infty}^{\infty} (uTv + vTu) dx = 0 \tag{T2}$$

where u, v have zero mean. We find that (T1), (T2) are necessary conditions on the T operator in order for (1) to have more than the standard number of conserved quantities. With these conditions the above evolution equations can be expected to be in the 1ST class. Condition (T2) follows from the fact that $T(x)$ is an odd function. To establish (T1) we use the representation (3c). Calling $\hat{T}_n = i \coth n\pi\delta/L$, and \hat{u}_n, \hat{v}_n , the Fourier coefficients of u, v respectively, assuming $\hat{u}_0 = \hat{v}_0 = 0$ (i.e., zero mean) and using the convolution theorem, then

$$\begin{aligned} T(uTv) &= \sum_{n=-\infty}^{\infty} \hat{T}_n \sum_{m=-\infty}^{\infty} \hat{T}_m \hat{v}_{n-m} \hat{u}_{n-m} \exp\left(\frac{in\pi x}{L}\right) \\ &= \sum_m \hat{u}_m \sum_n \hat{T}_n \hat{T}_{n-m} \hat{v}_{n-m} \exp\left(\frac{in\pi x}{L}\right) \\ &= -\sum_m \hat{u}_m \sum_n \hat{v}_{n-m} \exp\left(\frac{in\pi x}{L}\right) - \sum_m \hat{u}_m \sum_n \hat{T}_m \hat{T}_n \hat{v}_{n-m} \exp\left(\frac{in\pi x}{L}\right) \\ &\quad + \sum_m \hat{u}_m \sum_n \hat{T}_m \hat{T}_{n-m} \hat{v}_{n-m} \exp\left(\frac{in\pi x}{L}\right) \\ &= -u(x)v(x) - T(vTu) + (Tu)(Tv) \end{aligned} \tag{5}$$

where we have used the identity

$$\coth A \coth B = 1 + \coth(A - B)(-\coth A + \coth B).$$

The order of summation in (5) may be interchanged if $\sum |\hat{u}_n|, \sum |\hat{v}_n|$ exist. This establishes (T1).

The need for the T conditions can be seen from the following. The usual ILW equation on $(-\infty, \infty)$ was considered by Satsuma *et al* (1979). They showed that the derivations of the constants of motion and of the Bäcklund transformation do not depend on the specific kernel of T , so long as T satisfies conditions (T1) and (T2). This implies that these formulae will remain valid for equation (1) with (3) since the T conditions are satisfied. Actually one easily verifies (by differentiation) that the constants of motion given by Satsuma *et al* (1979) are also constants of motion of (1) with (3).

One may use the generalised Miura transformation, $u = \frac{1}{2}(\hat{\lambda} - V_x + iTV_x + \hat{\mu} e^{iV})$ to derive a generalisation of the so-called modified ILW equation. Namely if $\tilde{K}[u] = 0$ represents (1), and $\tilde{M}[v] = V_t + (1/\delta + \hat{\lambda})V_x + (TV)_{xx} + \hat{\mu}V_x e^{iV} + iV_x(TV)_x = 0$, then using the T conditions we have

$$\tilde{K}[u] = \theta \tilde{M}[v] \tag{6}$$

where

$$\theta = (-\partial_x + iT\partial_x + i\hat{\mu} e^{iV}).$$

Similarly it can be proven, by using the results of Fokas and Fuchssteiner (1980) (which also apply to the above Miura-type transformation) that (1) admits an auto-Bäcklund transformation if and only if (T1), (T2) are satisfied.

The 1st pair used by Kodama *et al* (1981) is also valid in the period problem. However, the simple compatibility argument does not indicate the importance of the T conditions. Namely, consider a linear scattering problem and a sequence of associated time evolutions of the form

$$i\psi_x^- + (u - \lambda)\psi^+ = \mu\psi^- \tag{7a}$$

$$\psi_t^\pm = Q_n^\pm \psi^\pm \quad n = 1, 2, \dots \tag{7b}$$

(7a) is to be thought of as a differential Riemann–Hilbert problem to find ψ^\pm (i.e., ψ^\pm are the boundary values of certain analytic functions) given an appropriate function $u(x)$. For each n , compatibility of (7a) and (7b) yields an evolution equation; namely, requiring $\psi_{xt}^\pm = \psi_{tx}^\pm$; and setting all coefficients of ψ^+ , ψ^- , ψ_x^- , ψ_{xx}^- , etc, to zero after using (7a) to eliminate derivatives of ψ^+ , gives an algorithmic procedure to determine compatible equations. The equivalent to the first two equations of the KdV hierarchy are obtained as follows. First, $Q_1^\pm = \partial_x + A^\pm$, whereupon we find $A^+ - A^- = 0$. Taking $A^\pm = A_0 = \text{constant}$, the compatible evolution equation is $u_t = u_x$. Second, $Q_2^\pm = i\partial_x^2 + iB^\pm\partial_x + iA^\pm$, we find $B^+ - B^- = 0$, $A^+ - A^- = -2iu_x$. Taking $B^\pm = B_0 = i(2\lambda + 1/\delta) = \text{constant}$, $A^\pm = \pm iu_x - (Tu)_x$, the compatible evolution equation is (1), without need for the T conditions. The underlying reason why such conditions must be added, and whether, in fact, (3) is the most general singular integral operator satisfying (T1), (T2) are open questions. Matsuno (1980) has given a different algorithm to derive such a hierarchy of equations.

Next we consider some special solutions of (1) with (3). For a wave of permanent form, $\partial_t \rightarrow -c\partial_x$, and (1) may be integrated once to

$$(\delta^{-1} - c)u + u^2 + (Tu)_x + A = 0 \tag{8}$$

with A constant. It has been claimed (Joseph and Egri 1978, Chen and Lee 1979) that

$$u(x - ct) = \frac{-(n\pi/L) \sinh(n\pi\delta/L)}{\cosh(n\pi\delta/L) + \cos[n\pi(x - ct)/L]} \tag{9a}$$

is a solution of (8). Its Fourier series representation is

$$u(\xi) = -\frac{n\pi}{L} \left[1 + 2 \sum_1^\infty (-1)^m \exp\left(-\frac{mn\pi\delta}{L}\right) \cos\left(\frac{mn\pi\xi}{L}\right) \right]. \tag{9b}$$

That (9) does not solve (8) may be seen by computing $\partial u/\partial \delta$ from (9a) and (9b), and Tu from (3d), because, as the reader may verify by direct calculation: $\partial u/\partial \delta = -\partial(Tu)/\partial x$. The correct solution is

$$u(x - ct) = \frac{-(n\pi/L) \sinh(n\pi\delta/L)}{\cosh(n\pi\delta/L) + \cos[(n\pi/L)(x - ct - i\phi)]} \tag{10a}$$

where $\phi < -\delta$. Its Fourier series is

$$u(\xi) = \frac{2n\pi}{L} \sum_1^\infty (-1)^m \sinh\left(\frac{mn\pi\delta}{L}\right) \exp\left(\frac{imn\pi(\xi - i\phi)}{L}\right) \tag{10b}$$

and its validity may be verified by computing $\partial u/\partial \delta$ from (10a) and (10b). This solution

is complex valued. (The Fourier representations in (9b) and (10b) differ because the transformation $x \rightarrow x - i\phi$, $\phi < \delta$, moves ξ outside the radius of convergence of (9b).)

More generally, periodic solutions may be obtained systematically via Hirota's method, revising slightly the methods of Chen and Lee (1979), Nakamura and Matsuno (1980) and Satsuma and Ablowitz (1980). Let $f^\pm(x) \equiv f(x \mp i\delta)$, let $f(x)$ be periodic ($2L$) and let $f(z)$ be analytic in the rectangle: $-L \leq \text{Re}(z) \leq L$, $-\delta \leq \text{Im}(z) \leq \delta$. Then by integrating $\int \tilde{T}(x - \zeta + i\delta)f(\zeta)d\zeta$ around this rectangle, one finds that

$$\frac{1}{2L} \int_{-L}^L \tilde{T}(x - z; \delta, L)[f^+(z) - f^-(z)] dz = i[f^+(x) + f^-(x)] + J_0 \tag{11}$$

where J_0 is an unimportant constant. It follows that if

$$u(x) = i[f^-(x) - f^+(x)] \tag{12}$$

where

$$f^\pm(x) = \partial_x(\ln F^\pm),$$

and if $f(x)$ is properly analytic, then (1) with (3) becomes a bilinear equation

$$(iD_t + i\delta^{-1}D_x - D_x^2 + A)F^+ \cdot F^- = 0 \tag{13}$$

with the usual notation (e.g. $D_x a.b \equiv (\partial_x - \partial_{x'})a(x)b(x')|_{x'=x}$), and $A = A(t)$. We emphasise that only those solutions of (13) that are analytic in the rectangle yield periodic solutions of (1), a fact that was overlooked previously.

The simplest real-valued solution of (1) with (3) was given by Nakamura and Matsuno (1980). It may be written in the form

$$u(\xi; m) = \frac{iK(m)}{L} \left[Z\left(\frac{K(m)}{L}(\xi - i\delta); m\right) - Z\left(\frac{K(m)}{L}(\xi + i\delta); m\right) \right] \tag{14a}$$

where $\xi = x - ct + x_0$, and m is the modulus. The analyticity condition is that

$$\delta/L < K'(m)/K(m) \tag{14b}$$

or, because $\delta/L = K'(k)/K(k)$, that the modulus of T exceeds the modulus of the solution. The wave speed is a complicated function of m , δ and L , and is given implicitly by Nakamura and Matsuno (1980).

Finally, we note that (13) is not the only 'bilinear' equation that may be obtained from (1), (3). For example, let $f(z)$ have a real period ($2L$), and be analytic in $-L \leq \text{Re}(z) \leq L$, $-\delta \leq \text{Im}(z) \leq \delta$ except for two poles at $z = z_0$ and z_0^* ($0 < \text{Im}(z_0) < \delta$) with residues b and b^* . Then the integral of $\int \tilde{T}(x - \zeta + i\delta)f(\zeta)d\zeta$ around the rectangle yields

$$\begin{aligned} \frac{1}{2L} \int_{-L}^L \tilde{T}(x - z)[f^+(z) - f^-(z)] dz &= i[f^+(x) + f^-(x)] + J_0 \\ -\frac{\pi i}{L} [b\tilde{T}(x - z_0 + i\delta) + b^*\tilde{T}(x - z_0^* + i\delta)] &\tag{15} \end{aligned}$$

instead of (11). In this case, (12) changes (1) with (3) into

$$\{iD_t + (i/\delta)D_x - D_x^2 + A + (\pi/L)[b\partial_x\tilde{T}(x - z_0 + i\delta) + b^*\partial_x\tilde{T}(x - z_0^* + i\delta)]\}F^+ \cdot F^- = 0. \tag{16}$$

The analyticity requirement is that $F(z)$ should be analytic in the usual rectangle except for simple branch points at z_0 and z_0^* , so that f^z has poles. We have not determined whether (15) yields any solutions of (1), (3) that are not available via (13).

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References

- Benjamin T B 1967 *J. Fluid Mech.* **29** 559
 Byrd P F and Friedman M D 1971 *Handbook of Elliptic Integrals for Scientists and Engineers* (New York: Springer)
 Chen H H and Lee Y C 1979 *Phys. Rev. Lett.* **43** 264
 Fokas A S and Fuchssteiner B 1980 *Bäcklund Transformations for Hereditary Symmetries, Nonlinear Analysis, TMA*
 Gel'fand I M and Shilov G 1964 *Generalized Functions, Properties and Operations* vol 1 (New York: Academic)
 Joseph R I 1977 *J. Phys. A: Math. Gen.* **10** L225
 Joseph R I and Egri R 1978 *J. Phys. A: Math. Gen.* **11** L97
 Kodama Y, Satsuma J and Ablowitz M J 1981 *Phys. Rev. Lett.* **46** 687
 Kubota T, Ko D R S and Dobbs D 1978 *J. Hydronaut.* **12** 157
 Matsuno Y 1980 *J. Phys. Soc. Japan* **48** 1024
 Nakamura A and Matsuno Y 1980 *J. Phys. Soc. Japan* **48** 653
 Ono H 1975 *J. Phys. Soc. Japan* **39** 1082
 Satsuma J and Ablowitz M J 1980 in *Nonlinear Partial Differential Equations in Engineering and Applied Science* ed R L Sternberg, A J Kalinowki and J S Papadakis (New York: Marcel Dekker) p 397
 Satsuma J, Ablowitz M J and Kodama Y 1979 *Phys. Lett.* **73A** 283